

Analytical calculation of optimal POVM for unambiguous discrimination of quantum states using KKT method

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Abstract

In the present paper, an exact analytic solution for the optimal unambiguous state discrimination(OPUSD) problem involving an arbitrary number of pure linearly independent quantum states with real and complex inner product is presented. Using semidefinite programming and Karush-Kuhn-Tucker convex optimization method, we derive an analytical formula which shows the relation between optimal solution of unambiguous state discrimination problem and an arbitrary number of pure linearly independent quantum states.

Keywords: Unambiguous discrimination, Semidefinite programming, linearly independent States.

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1 Introduction

Many applications in quantum communication and quantum cryptography are based on transmitting quantum systems that, with given prior probabilities, are prepared in one from a set of known mutually nonorthogonal states[1]. A fundamental aspect of quantum information theory is that nonorthogonal quantum states cannot be perfectly distinguished. Therefore, a central problem in quantum mechanics is to design measurements optimized to distinguish between a collection of nonorthogonal quantum states. The topic of quantum state discrimination was firmly established in the 1970s by the pioneering work of Helstrom [2], who considered a minimum error discrimination of two known quantum states. In this case, the state identification is probabilistic. Another possible discrimination strategy is the so-called unambiguous state discrimination (USD) where the states are successfully identified with nonunit probability, but without error.

USD was originally formulated and analyzed by Ivanovic, Dieks, and Peres [3, 4, 5] in 1987. The solution for unambiguous discrimination of two known pure states appearing with arbitrary prior probabilities was obtained by Jaeger and Shimony [6]. Although the two-state problem is well developed, the problem of unambiguous discrimination between multiple quantum states has received considerably less attention. The problem of discrimination among three nonorthogonal states was first considered by Peres and Terno [5]. They developed a geometric approach and applied it numerically on several examples. A different method was considered by Duan and Guo [7] and Sun et al.[8]. Chefles [9] showed that a necessary and sufficient condition for the existence of unambiguous measurements for distinguishing between N quantum states is that the states are linearly independent. He also proposed a simple suboptimal measurement for unambiguous discrimination for which the probability of an inconclusive result is the same regardless of the state of the system. Equivalently, the measurement yields an equal probability of correct detection of each one of the ensemble states.

Over the past years, semidefinite programming (SDP) has been recognized as a valuable numerical tool for control system analysis and design. In SDP, one minimizes a linear function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. SDP has been studied (under various names) as far back as the 1940s. Subsequent research in semidefinite programming during the 1990s was driven by applications in combinatorial optimization [10], communications and signal processing [11, 12, 13] and other areas of engineering [14]. Although semidefinite programming is designed to be applied in numerical methods, it can be used for analytic computations, too. In the context of quantum computation and quantum information, Barnum, Saks, and Szegedy have reformulated quantum query complexity in terms of a semidefinite program [15]. The problem of finding the optimal measurement to distinguish between a set of quantum states was first formulated as a semidefinite program in 1974 by Holevo, who gave optimality conditions equivalent to the complementary slackness conditions[2]. Recently, Eldar, Megretski, and Verghese showed that the optimal measurements can be found efficiently by solving the dual followed by the use of linear programming[16]. Also in Ref. [17], SDP has been used to show that the standard algorithm implements the optimal set of measurements. All of the above mentioned applications indicate that the method of SDP is very useful. The reason why the area has shown relatively slow progress until recently within the rapidly evolving field of quantum information is that it poses quite formidable mathematical challenges. Except for a handful of very special cases, no general exact solution has been available involving more than two arbitrary states and mostly numerical algorithms are proposed for finding optimal measurements for quantum-state discrimination, where the theory of the semidefinite programming provides a simple check of the optimality of the numerically obtained results. In Ref. [18] we obtained the feasible region in terms of the inner product of the reciprocal states and the feasible region in terms of the inner product of the states which enables us to solve the problem without using reciprocal states. Moreover, for the real inner product of states, we obtained an exact analytic solution for

OPUSD problem involving an arbitrary number of pure linearly independent quantum states by using KKT convex optimization method. In this paper, an exact analytic solution for the optimal unambiguous state discrimination(OPUSD) problem involving an arbitrary number of pure linearly independent quantum states with real and complex inner product of states, using the Karush- Kuhn-Tucker convex optimization method, is given.

The organization of the paper is as follows. First, the definition of the unambiguous quantum state discrimination is given. Then, Semidefinite programming, Karush-Kuhn-Tucker (KKT) theorem and SDP formulation of unambiguous discrimination is studied. Finally for the real and complex inner product of reciprocal states, an exact analytic solution for OPUSD problem involving an arbitrary number of pure linearly independent quantum states is presented by using KKT convex optimization method. The paper ends with a brief conclusion.

2 Unambiguous quantum state discrimination

In quantum theory, measurements are represented by positive operator valued measures (POVMs). A measurement is described by a collection M_k of measurement operators. These operators are acting on the state space of the system being measured. The index k refers to the measurement outcomes that may occur in the experiment. In quantum information theory the measurement operators M_k are often called Kraus operators [19]. If we define the operator

$$\Pi_k = M_k^\dagger M_k, \quad (2.1)$$

the probability of obtaining the outcome k for a given state ρ_i is given by $p(k|i) = \text{Tr}(\Pi_k \rho_i)$. Thus, the set of operators Π_k is sufficient to determine the measurement statistics.

2.1 Definition of the POVM

A set of operators $\{\Pi_k\}$ is named a positive operator valued measure if and only if the following two conditions are met: (1) each operator Π_k is positive positive $\Leftrightarrow \langle \psi | \Pi_k | \psi \rangle \geq 0, \forall |\psi\rangle$ and

(2) the completeness relation is satisfied, i.e.,

$$\sum_k \Pi_k = 1. \quad (2.2)$$

The elements of $\{\Pi_k\}$ are called effects or POVM elements. On its own, a given POVM $\{\Pi_k\}$ is enough to give complete knowledge about the probabilities of all possible outcomes; measurement statistics is the only item of interest. Consider a set of known states $\rho_i, i = 1, \dots, N$ with their prior probabilities η_i . We are looking for a measurement that either identifies a state uniquely (conclusive result) or fails to identify it (inconclusive result). The goal is to minimize the probability of inconclusive results. The measurements involved are typically generalized measurements. A measurement described by a POVM $\{\Pi_k\}_{k=1}^N$ is called an unambiguous state discrimination measurement (USD) on the set of states $\{\rho_i\}_{i=1}^N$ if and only if the following conditions are satisfied. (1) The POVM contains the elements $\{\Pi_k\}_{k=0}^N$ where N is the number of different signals in the set of states. The element Π_0 is related to an inconclusive result, while the other elements correspond to an identification of one of the states $\rho_i, i = 1, \dots, N$. (2) No states are wrongly identified, that is, $\text{Tr}(\rho_i \Pi_k) = 0 \forall i \neq k, k = 1, \dots, N$. Each USD measurement gives rise to a failure probability, that is, the rate of inconclusive result. This can be calculated as

$$Q = \sum_i \eta_i \text{Tr}(\eta_i \Pi_0). \quad (2.3)$$

The success probability can be calculated as

$$P = 1 - Q = \sum_i \eta_i \text{Tr}(\eta_i \Pi_i). \quad (2.4)$$

A measurement described by a POVM $\{\Pi_k^{opt}\}$ is called an optimal unambiguous state discrimination measurement (OUSDM) on a set of states $\{\rho_i\}$ with the corresponding prior probabilities $\{\eta_i\}$ if and only if the following conditions are satisfied. (1) The POVM $\{\Pi_k^{opt}\}$ is a USD measurement on $\{\rho_i\}$. (2) The probability of inconclusive result is minimal, that is, $Q(\{\Pi_k^{opt}\}) = \min Q(\{\Pi_k\})$, where the minimum is taken over all USD. Unambiguous state

discrimination is an error-free discrimination. This implies a strong constraint on the measurement. Suppose that a quantum system is prepared in a pure quantum state drawn from a collection of given states $\{|\psi_i\rangle\}, 1 \leq i \leq N$ in d -dimensional complex Hilbert space \mathcal{H} with $d \geq N$. These states span a subspace \mathcal{U} of \mathcal{H} . In order to detect the state of the system, a measurement is constructed comprising $N + 1$ measurement operators $\{\Pi_i, 0 \leq i \leq N\}$. Given that the state of the system is $|\psi_i\rangle$, the probability of obtaining outcome k is $\langle\psi_i|\Pi_k|\psi_i\rangle$. Therefore, in order to ensure that each state is either correctly detected or an inconclusive result is obtained, we must have

$$\langle\psi_i|\Pi_k|\psi_i\rangle = p_i\delta_{ij}, \quad 1 \leq i, k \leq N \quad (2.5)$$

for some $0 \leq p_i \leq 1$. Since $\Pi_0 = I_d - \sum_{i=1}^N \Pi_i$, we have $\langle\psi_i|\Pi_0|\psi_i\rangle = 1 - p_i$. So a system with given state $|\psi_i\rangle$, the state of the system is correctly detected with probability p_i and an inconclusive result is obtained with probability $1 - p_i$. It was shown in Ref. [20] that Eq.(2.5) is satisfied if and only if the vectors $|\psi_i\rangle$ are linearly independent, or equivalently, $\dim \mathcal{U} = N$. Therefore, we will take this assumption throughout the paper. In this case, we may choose [21]

$$\Pi_i = p_i|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|, \quad 1 \leq i \leq N, \quad (2.6)$$

where the vectors $|\tilde{\psi}_i\rangle \in \mathcal{U}$ are the reciprocal states associated with the states $|\psi_i\rangle$, i.e., there are unique vectors in \mathcal{U} such that

$$\langle\tilde{\psi}_k|\psi_i\rangle = \delta_{ij}, \quad 1 \leq i, k \leq N. \quad (2.7)$$

With Φ and $\tilde{\Phi}$ we denote the matrices such that their columns are $|\psi_i\rangle$ and $|\tilde{\psi}_i\rangle$, respectively. Then, one can show that $\tilde{\Phi}$ is

$$\tilde{\Phi} = \Phi(\Phi^*\Phi)^{-1}. \quad (2.8)$$

Since the vectors $|\psi_i\rangle, i = 1, \dots, N$ are linearly independent, $\Phi^*\Phi$ is always invertible. Alternatively,

$$\tilde{\Phi} = (\Phi\Phi^*)^\dagger\Phi, \quad (2.9)$$

so that

$$|\tilde{\psi}_i\rangle = (\Phi\Phi^*)^\dagger |\psi_i\rangle, \quad (2.10)$$

where $(\dots)^\dagger$ denotes the Moore-Penrose pseudoinverse [22, 23]. The inverse is taken on the subspace spanned by the columns of the matrix. If the state $|\psi_i\rangle$ is prepared with prior probability η_i , then the total probability of correctly detecting the state is

$$P = \sum_{i=1}^N \eta_i \langle \psi_i | \Pi_i | \psi_i \rangle = \sum_{i=1}^N \eta_i p_i \quad (2.11)$$

and the probability of the inconclusive result is given by

$$Q = 1 - P = \sum_{i=1}^N \eta_i \langle \psi_i | \Pi_0 | \psi_i \rangle = 1 - \sum_{i=1}^N \eta_i p_i \quad (2.12)$$

In general, an optimal measurement for a given strategy depends on the quantum states and the prior probabilities of their appearance. In the unambiguous discrimination for a given strategy and a given ensemble of states, the goal is to find a measurement which minimizes the inconclusive result. In fact, it is known that USD (of both pure and mixed states) is a convex optimization problem. Mathematically, this means that the quantity which is to be optimized as well as the constraints on the unknowns, are convex functions. Practically, this implies that the optimal solution can be computed in an extremely efficient way. This is therefore a very useful tool. Nevertheless, our aim is to understand the structure of USD in order to relate it with neat and relevant quantities and to find feasible region for numerical and analytic solutions.

3 Semidefinite programming

A SDP problem requires minimizing a linear function subject to a linear matrix inequality (LMI) constraint

$$\mathbf{minimize} \ p = c^T x, \text{ such that } F(x) \geq 0, \quad (3.13)$$

where c^T is a given vector, $x = (x_1, \dots, x_n)$, and $F(x) = F_0 + \sum_i x_i F_i$, for some fixed Hermitian matrices F_i . The inequality sign in $F(x) \geq 0$ means that $F(x)$ is positive semidefinite. This problem is called the primal problem. Vectors x whose components are the variables of the problem and satisfy the constraint $F(x) \geq 0$ are called primal feasible points, and if they satisfy $F(x) > 0$ they are called strictly feasible points. The minimal objective value $c^T x$ is by convention denoted by P^* and is called the primal optimal value. Due to the convexity of set of feasible points, SDP has a nice duality structure with the associated dual program being

$$\mathbf{maximize} \quad -\text{Tr}[F_0 Z], \quad Z \geq 0, \quad \text{Tr}[F_i Z] = c_i. \quad (3.14)$$

Here the variable is the real symmetric (or Hermitian) matrix Z , and the data c, F_i are the same as in the primal problem. Correspondingly, matrix Z satisfying the constraints are called dual feasible (or strictly dual feasible if $Z > 0$). The maximal objective value of $-\text{Tr}[F_0 Z]$, i.e., the dual optimal value, is denoted by d^* . The objective value of a primal (dual) feasible point is an upper (lower) bound on $P^*(d^*)$. The main reason why one is interested in the dual problem is that one can prove that $d^* \leq P^*$, and under relatively mild assumptions, we can have $P^* = d^*$. If the equality holds, one can prove the following optimality condition on x . A primal feasible x and a dual feasible Z are optimal which is denoted by \hat{x} and \hat{Z} if and only if

$$F(\hat{x})\hat{Z} = \hat{Z}F(\hat{x}). \quad (3.15)$$

This latter condition is called the complementary slackness condition. In one way or another, numerical methods for solving SDP problems always exploit the inequality $d \leq d^* \leq P^* \leq P$, where d and P are the objective values for any dual feasible point and primal feasible point, respectively. The difference

$$P^* - d^* = c^T x + \text{Tr}[F_0 Z] = \text{Tr}[F_x Z] \geq 0 \quad (3.16)$$

is called the duality gap. If the equality holds $d^* = P^*$, i.e., the optimal duality gap is zero, then we say that strong duality holds.

3.1 Karush-Kuhn-Tucker (KKT) theorem

Assuming that functions g_i , h_i are differentiable and that strong duality holds, there exists vectors $\xi \in R^k$ and $y \in R^m$ such that the gradient of dual Lagrangian $L(x^*, \xi^*, y^*) = f(x^*) + \sum_i \xi_i^* h_i(x^*) + \sum_i y_i^* g_i(x^*)$ over x vanishes at x^* :

$$h_i(x^*) = 0 \text{ (primal feasible),}$$

$$g_i(x^*) \leq 0 \text{ (primal feasible),}$$

$$y_i^* \geq 0 \text{ (dual feasible), } y_i^* g_i(x^*) = 0,$$

$$\nabla f(x^*) + \sum_i \xi_i^* \nabla h_i(x^*) + \sum_i y_i^* \nabla g_i(x^*) = 0. \quad (3.17)$$

Then x^* and (ξ_i^*, y_i^*) are primal and dual optimal with zero duality gap. In summary, for any convex optimization problem with differentiable objective and constraint functions, the points which satisfy the KKT conditions are primal and dual optimal, and have zero duality gap. Necessary KKT conditions satisfied by any primal and dual optimal pair and for convex problems, KKT conditions are also sufficient. If a convex optimization problem with differentiable objective and constraint functions satisfies Slaters condition, then the KKT conditions provide necessary and sufficient conditions for optimality: Slaters condition implies that the optimal duality gap is zero and the dual optimum is attained, so x is optimal if and only if there are (ξ_i^*, y_i^*) such that they, together with x , satisfy the KKT conditions.

3.2 Slaters condition

Suppose x^* solves

$$\textbf{minimize } f(x) \text{ s.t. } g_i(x) \geq b_i, \quad i = 1, \dots, m \quad (3.18)$$

and the feasible set is nonempty. Then there is a nonnegative vector ξ such that for all x

$$L(x, \xi) = f(x) + \xi^T [b - g(x)] \leq f(x^*) = L(x^*, \xi). \quad (3.19)$$

In addition, if $f(\dots)$, $g_i(\dots)$, $i = 1, \dots, m$, are continuously differentiable, then

$$\frac{\partial f(x^*)}{\partial(x_j)} - \xi \frac{\partial g(x^*)}{\partial(x)} = 0. \quad (3.20)$$

In the spatial case the vector x is a solution of the linear program

$$\textbf{minimize } c^T x, \text{ such that } Ax = b, x \geq 0, \quad (3.21)$$

if and only if there exist vectors $\xi \in R^k$ and $y \in R^m$ for which the following conditions hold for $(x, \xi, y) = (x^*, \xi^*, y^*)$:

$$A^T \xi + y = c, Ax = b, x_i \geq 0; y \geq 0;$$

$$x_i y_i = 0, \quad i = 1, \dots, m. \quad (3.22)$$

A solution (x^*, ξ^*, y^*) is called strictly complementary, if $x^* + y^* > 0$, i.e., if there exists no index $i \in 1, \dots, m$ such that $x_i^* = y_i^*$.

4 SDP Formulation of unambiguous discrimination

Eldar, Megretski, and Verghese in Ref.[24] have showed that the unambiguous discrimination problem can be reduced to SDP method and the KKT conditions can be defined as

$$\begin{aligned} F(p) &= \sum_{i=1}^N p_i F_i + F_0 \geq 0, \quad Tr(\Pi_i X) = z_i + \eta_i, \\ z_i &\geq 0, 1 \leq i \leq N, \quad AF(p) = 0, \quad X(I_d - \sum_{i=1}^N p_i \Pi_i) = 0, \\ z_i p_i &= 0, \quad 1 \leq i \leq N, \quad \exists p_i : \sum_{i=1}^N p_i F_i + F_0 \geq 0, \\ \exists X, z &: X \geq 0, z \geq 0 \end{aligned} \quad (4.23)$$

such that

$$\begin{aligned}
|p\rangle &=: \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}, |c\rangle =: \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_2 \end{pmatrix}, \\
F_0 &=: \begin{pmatrix} I_d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, F_1 =: \begin{pmatrix} -\Pi_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, F_2 =: \begin{pmatrix} -\Pi_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \dots, \\
F_N &=: \begin{pmatrix} -\Pi_N & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, A =: \begin{pmatrix} X_d & Y \\ Y^T & \begin{pmatrix} z_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & z_N \end{pmatrix} \end{pmatrix}, \quad (4.24)
\end{aligned}$$

where X_d is the $d \times d$ matrix and Y is the $N \times d$ matrix.

5 Analytical solution of unambiguous state discrimination problem

When the set of states $\{\psi_i\}_{i=1}^N$ and the prior probabilities $\{\eta_i\}_{i=1}^N$ are given, the POVM elements for the measurement ($\Pi_i = \sum_{i=1}^N p_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$, $\Pi_0 = I - \sum_{i=1}^N \Pi_i$) depend on only the variables p_i and these variables lie in the feasible region. Feasible region that gives the domain of acceptable values of p_i is determined by the following equation [25, 26, 27, 28]:

$$1 - \sum_i D_i p_i + \sum_{i < j} D_{ij} p_i p_j - \sum_{i < j < k} D_{ijk} p_i p_j p_k + \dots + (-1)^N \sum_{i_1 < i_2 < \dots < i_N} p_{i_1} p_{i_2} \dots p_{i_N} D_{i_1 i_2 \dots i_N} = 0, \quad (5.25)$$

where the set of $\{D_{i_1 i_2 \dots i_N}\}$ are the subdeterminants (minor) of matrix D defined by

$$D = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1N} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{N1} & \tilde{a}_{N2} & \cdots & \tilde{a}_{NN} \end{pmatrix} \quad (5.26)$$

with $\tilde{a}_{ij} = \langle \tilde{\psi}_i | \tilde{\psi}_j \rangle$.

In this section, for the real and complex inner product of reciprocal states, an exact analytic solution for OPUSD problem involving an arbitrary number of pure linearly independent quantum states is presented by using KKT convex optimization method. The KKT conditions for unambiguous discrimination of N linearly independent states are given by

$$\begin{aligned} I - \sum_{i=1}^N p_i |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i| &\geq 0, \quad p_i \geq 0, \\ (I - \sum_{i=1}^N p_i |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|)X &= X(I - \sum_{i=1}^N p_i |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|) = 0, \\ z_i p_i &= 0, \\ \text{Tr}(X |\tilde{\psi}_i\rangle \langle \tilde{\psi}_i|) &= z_i + \eta_i, \quad \eta_i \geq 0, \quad i = 1, \dots, N \end{aligned} \quad (5.27)$$

Then using KKT conditions one can show that

$$\begin{pmatrix} 1 - p_1 \tilde{a}_{11} & -p_2 \tilde{a}_{12} & \cdots & -p_N \tilde{a}_{1N} \\ -p_1 \tilde{a}_{21} & 1 - p_2 \tilde{a}_{22} & \cdots & -p_N \tilde{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -p_1 \tilde{a}_{N1} & -p_2 \tilde{a}_{N2} & \cdots & 1 - p_N \tilde{a}_{NN} \end{pmatrix} \times \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NN} \end{pmatrix} = 0 \quad (5.28)$$

If $p_i \geq 0, (i = 1, \dots, N)$ using Eq.(5.27) we can conclude that X is the rank one matrix and we have

$$x_{ij} = e^{i\varphi_i} e^{i\varphi_j} \sqrt{\eta_i} \sqrt{\eta_j} \quad (5.29)$$

Consequently, the Eq. (5.28) can be written as

$$\begin{pmatrix} 1 - p_1 \tilde{a}_{11} & -p_2 \tilde{a}_{12} & \cdots & -p_N \tilde{a}_{1N} \\ -p_1 \tilde{a}_{21} & 1 - p_2 \tilde{a}_{22} & \cdots & -p_N \tilde{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -p_1 \tilde{a}_{N1} & -p_2 \tilde{a}_{N2} & \cdots & 1 - p_N \tilde{a}_{NN} \end{pmatrix} \times \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1N} \end{pmatrix} = 0 \quad (5.30)$$

Or, equivalently,

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1N} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{N1} & \tilde{a}_{N2} & \cdots & \tilde{a}_{NN} \end{pmatrix} \times \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{NN} \end{pmatrix} \times \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N1} \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1N} \end{pmatrix} \quad (5.31)$$

If the first condition in Eq. (5.27) is satisfied, using above equation one can show that the optimal solution for p_i is given by

$$p_i = \frac{1}{e^{i\varphi_i} \sqrt{\eta_i} \det(D)} \times \det \begin{pmatrix} \tilde{a}_{1,1} & \cdots & \tilde{a}_{1,i-1} & e^{i\varphi_1} \sqrt{\eta_1} & \tilde{a}_{1,i+1} & \cdots & \tilde{a}_{1,N} \\ \tilde{a}_{2,1} & \cdots & \tilde{a}_{2,i-1} & e^{i\varphi_2} \sqrt{\eta_2} & \tilde{a}_{2,i+1} & \cdots & \tilde{a}_{2,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{N,1} & \cdots & \tilde{a}_{N,i-1} & e^{i\varphi_i} \sqrt{\eta_i} & \tilde{a}_{N,i+1} & \cdots & \tilde{a}_{N,N} \end{pmatrix} \quad (5.32)$$

Using the fact that the matrix whose components are \tilde{a}_{ij} is inverse matrix of the one whose components are $a_{ij} = \langle \psi_i | \psi_j \rangle$, we have

$$\begin{pmatrix} x_{11} p_1 \\ x_{12} p_2 \\ \vdots \\ x_{1N} p_N \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1N} \end{pmatrix} \Rightarrow x_{ii} p_i = \sum_{j=1}^N a_{ij} x_{ij}$$

Using Eq. (5.29) we have

$$p_i = \sum_{j=1}^N \frac{e^{i(\varphi_i + \varphi_j)}}{e^{2i\varphi_i}} \sqrt{\frac{\eta_i}{\eta_j}} a_{ij}$$

Then we can rewrite the probability p_i in a simpler form

$$p_i = e^{-i\varphi_i} \sum_{j=1}^N e^{i\varphi_j} \sqrt{\frac{\eta_i}{\eta_j}} a_{ij} \quad (5.33)$$

where $e^{-i\varphi_j} a_{ij}$ is determined such that p_i satisfy the condition $0 \leq p_i \leq 1$. Then the success probability with respect to \tilde{a}_{ij} components is given by

$$P = -\frac{1}{\det(D)} \times \det \begin{pmatrix} 0 & e^{i\varphi_1} \sqrt{\eta_1} & \cdots & e^{i\varphi_N} \sqrt{\eta_N} \\ e^{i\varphi_1} \sqrt{\eta_1} & & & \\ \vdots & & D & \\ e^{i\varphi_N} \sqrt{\eta_N} & & & \end{pmatrix} \quad (5.34)$$

The success probability with respect to a_{ij} components is as follows

$$P = \sum_{i=1}^N \eta_i p_i = \sum_{i,j=1}^N e^{i(\varphi_i - \varphi_j)} \sqrt{\eta_i \eta_j} a_{ij} = \left\| \sum_{j=1}^N \sqrt{\eta_j} e^{i\varphi_j} |\psi_j\rangle \right\|^2 \quad (5.35)$$

If the first condition of KKT is not satisfied, for a specified i , p_i does not lie in the feasible region ($p_i \leq 0$ or $p_i \geq 1$). In this case, one can omit the j th row and j th column in the square $N \times N$ matrices and j th row in the row matrices in Eqs. (5.32) and (5.34), and find the optimal solutions with $p_i = 0$. After this reduction, if any other p_i does not lie in the feasible region, the same procedure will be repeated. Noted that a similar result was derived in Ref. [29] with another method. The equations (5.33) and (5.35) gives an analytical relation between the maximum average success probability and the N pure linearly independent quantum states to be discriminated. In general, finding an exact analytic solution for the OPUSD problem involving an arbitrary number of pure linearly independent quantum states is hard, since the explicit expressions of the phases $e^{i\varphi_j}$ ($j = 1, \dots, N$) are not given in equations (5.33) and (5.35). However using (5.33) and (5.35) one can simplify the calculation of the optimal solution in special cases and it may also drive some bounds for the average success probability [18, 29]. Then, the approximated methods are useful for unambiguous discrimination of N linearly independent quantum states. Since we have presented an analytic relation for the feasible

region of N linearly independent quantum states, and this region is convex, then one can easily obtain the optimal POVM by some well-known numerical methods such as constrained linear or nonlinear least-squares, interior points, and simplex and quadratic programming methods [30].

6 Conclusion

In conclusion, for the real and complex inner product of states, we have been able to obtain an exact analytic solution for OPUSD problem involving an arbitrary number of pure linearly independent quantum states by using KKT convex optimization method. Moreover, Using semidefinite programming and Karush-Kuhn-Tucker convex optimization method, we have been able to obtain an analytical formula which shows the relation between optimal solution of unambiguous discrimination problem and an arbitrary number of pure linearly independent quantum states to be identified. Using this analytical formula one can simplify the calculation of the optimal solution in special cases and it may also drive some bounds for the average success probability.

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